If the complementary work is regarded as a functional of both the Piola stress tensor and the vector of displacements, the meaning of Castigliano's principle is lost as a variational principle, which selects among all statically possible states of stress those which satisfy the conditions of continuity.

In paper [4] the complementary work is treated as a functional of the Piola stress tensor only and it is established that continuity equations (9) follow from the stationary state of complementary work. Nevertheless, the question about the possibility of expressing the gradients of displacements and the specific complementary strain work in terms of components of the Piola stress tensor remains open in paper [4].

In fact, as was shown above, the gradients of displacements and the specific complementary strain work can be represented as a function of the components of the Piola stress tensor only. Therefore the principle of Castigliano which was formulated for the Piola stress tensor retains its significance also in the nonlinear theory of elasticity.

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ON PLANE CONTACT PROBLEMS OF THE THEORY OF ELASTICITY IN THE PRESENCE OF ADHESION OR FRICTION

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As is known, plane contact problems of the theory of elasticity for a half-plane in the presence of adhesion or friction in the contact domain have been studied sufficiently well.

Corresponding contact problems for elastic solids different in shape or their mechanical properties from an isotropic elastic half-plane, have begun to be worked upon comparatively recently. The papers of Popov [1, 2] should here be singled out first.

A general analysis of the structure of the solution of nonclassical plane contact problems in the presence of adhesion or friction in the contact domain is given herein. Possible methods of solving them effectively are indicated.

1. Mathematical formulation. Some auxiliary results. We call the following the nonclassical mixed problems: (1) mixed problems of elasticity theory for bodies of complex shape (strip, layer, circle, sphere, infinite cylinder, wedge, etc.),

2) dynamical mixed problems of elasticity theory, (3) three-dimensional problems of elasticity theory for stamps of complex planform, (4) mixed problems of couple-stress elasticity theory, (5) mixed problems of linear viscoelasticity, etc.

In the majority of cases of practical importance, the listed mixed problems in the plane modification can be reduced by operational calculus methods to the solution of the following three kinds of integral equations of the first kind:

a) In the presence of complete adhesion over the contact domain

$$\begin{gathered}\int_{-1}^{1} \varphi_1(\xi) K_{11}\left(\frac{\xi-x}{\lambda}\right) d\xi - \varepsilon \int_{-1}^{1} \varphi_2(\xi) K_{12}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_1(x) \\ \varepsilon \int_{-1}^{1} \varphi_1(\xi) K_{21}\left(\frac{\xi-x}{\lambda}\right) d\xi + \int_{-1}^{1} \varphi_2(\xi) K_{22}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_2(x)\end{gathered} \qquad \begin{pmatrix} |x| \leqslant 1 \\ K_{12}(t) = K_{21}(t) \end{pmatrix} \tag{1.1}$$

b) In the presence of friction forces in the contact domain

$$\int_{-1}^{1} \varphi_1(\xi) K_{11}\left(\frac{\xi-x}{\lambda}\right) d\xi - k\varepsilon \int_{-1}^{1} \varphi_1(\xi) K_{12}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_1(x) \qquad (|x| \leqslant 1) \tag{1.2}$$

c) In the absence of adhesion and friction in the contact domain

$$\int_{-1}^{1} \varphi_1(\xi) K_{11}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_1(x) \qquad (|x| \leqslant 1) \tag{1.3}$$

Here $\lambda \in (0, \infty)$, $\varepsilon \geqslant 0$, $k \geqslant 0$ are nondimensional parameters. The kernels $K_{jl}(t)$ $(j = 1, 2,\ l = 1, 2)$ can be represented for all $|t| = |\xi - x| \lambda^{-1} < \infty$ in the form

$$K_{jj}(t) = -\ln|t| + F_{jj}(t), \qquad K_{12}(t) = {}^1/_2\pi \operatorname{sgn} t + F_{12}(t) \tag{1.4}$$

where the functions $F_{jj}(t)$ are even, and $F_{12}(t)$ is odd in t, but they are all, at least, continuous. Later, additional constraints will be imposed on $F_{jl}(t)$. We consider the functions $f_j(x)$ to belong to the class $H_n^\alpha(-1, 1)$, $n \geqslant 1$, $0 < \alpha \leqslant 1$ (*).

Let us represent the functions $\varphi_j(x)$ as

$$\varphi_j(x) = \varphi_j^\circ(x) + \varphi_j^*(x) \tag{1.5}$$

Here $\varphi_j^\circ(x)$ satisfy the following respective integral equations:

$$\text{a)} \quad \int_{-1}^{1} \varphi_1^\circ(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right] d\xi - \frac{\pi\varepsilon}{2}\int_{-1}^{1} \varphi_2^\circ(\xi) \operatorname{sgn}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_1(x)$$

$$(|x| \leqslant 1) \tag{1.6}$$

$$\int_{-1}^{1} \varphi_2^\circ(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{22}(0)\right] d\xi + \frac{\pi\varepsilon}{2}\int_{-1}^{1} \varphi_1^\circ(\xi) \operatorname{sgn}\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f_2(x)$$

$$\text{b)} \quad \int_{-1}^{1} \varphi_1^\circ(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right] d\xi - \frac{\pi k\varepsilon}{2}\int_{-1}^{1} \varphi_1^\circ(\xi) \operatorname{sgn}\left(\frac{\xi-x}{\lambda}\right) d\xi =$$

$$= \pi f_1(x) \qquad (|x| \leqslant 1) \tag{1.7}$$

---

*) If $f(x) \in H_n^\alpha(-\beta, \beta)$, then its nth derivative satisfies the Hölder condition with exponent $0 < \alpha \leqslant 1$ for $x \in [-\beta, \beta]$.

$$\text{c)} \int_{-1}^{1} \varphi_1{}^\circ(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right]d\xi = \pi f_1(x) \qquad (|x| \leqslant 1) \tag{1.8}$$

On the basis of (1.1)−(1.8) we easily arrive at the deduction that the $\varphi_j{}^\circ(\xi)$, which satisfy (1.6)−(1.8), are the principal terms of the asymptotics of the functions $\varphi_j(\xi)$ for large values of the parameter λ.

The corrections $\varphi_j{}^*(\xi)$, which vanish for $\lambda \to \infty$, should be found from the following integral equations:

$$\text{a)} \int_{-1}^{1} \varphi_1{}^*(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right]d\xi - \frac{\pi\varepsilon}{2}\int_{-1}^{1} \varphi_2{}^*(\xi)\,\mathrm{sgn}\left(\frac{\xi-x}{\lambda}\right)d\xi = \pi f_1{}^*(x)$$

$$(|x| \leqslant 1) \tag{1.9}$$

$$\int_{-1}^{1} \varphi_2{}^*(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{22}(0)\right]d\xi + \frac{\pi\varepsilon}{2}\int_{-1}^{1} \varphi_1{}^*(\xi)\,\mathrm{sgn}\left(\frac{\xi-x}{\lambda}\right)d\xi = \pi f_2{}^*(x)$$

$$f_j{}^*(x) = -\frac{1}{\pi}\int_{-1}^{1} [\varphi_j{}^\circ(\xi) + \varphi_j{}^*(\xi)]\left[F_{jj}\left(\frac{\xi-x}{\lambda}\right) - F_{jj}(0)\right]d\xi \pm$$

$$\pm \frac{\varepsilon}{\pi}\int_{-1}^{1} [\varphi_j{}^\circ(\xi) + \varphi_j{}^*(\xi)]\,F_{12}\left(\frac{\xi-x}{\lambda}\right)d\xi \tag{1.10}$$

Here the plus is for $j = 1$ and the minus for $j = 2$.

$$\text{b)} \int_{-1}^{1} \varphi_1{}^*(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right]d\xi - \frac{\pi k\varepsilon}{2}\int_{-1}^{1} \varphi_1{}^*(\xi)\,\mathrm{sgn}\left(\frac{\xi-x}{\lambda}\right)d\xi = \pi f_1{}^*(x)$$

$$(|x| \leqslant 1) \tag{1.11}$$

$$f_1{}^*(x) = -\frac{1}{\pi}\int_{-1}^{1} [\varphi_1{}^\circ(\xi) + \varphi_1{}^*(\xi)]\left[F_{11}\left(\frac{\xi-x}{\lambda}\right) - F_{11}(0)\right]d\xi +$$

$$+ \frac{\varepsilon k}{\pi}\int_{-1}^{1} [\varphi_1{}^\circ(\xi) + \varphi_1{}^*(\xi)]\,F_{12}\left(\frac{\xi-x}{\lambda}\right)d\xi \tag{1.12}$$

$$\text{c)} \int_{-1}^{1} \varphi_1{}^*(\xi)\left[-\ln\left|\frac{\xi-x}{\lambda}\right| + F_{11}(0)\right]d\xi = \pi f_1{}^*(x) \qquad (|x| \leqslant 1) \tag{1.13}$$

$$f_1{}^*(x) = -\frac{1}{\pi}\int_{-1}^{1} [\varphi_1{}^\circ(\xi) + \varphi_1{}^*(\xi)]\left[F_{11}\left(\frac{\xi-x}{\lambda}\right) - F_{11}(0)\right]d\xi \tag{1.14}$$

The integral equations (1.6)−(1.8) can be represented as

$$\int_{-1}^{1} \varphi^\circ(\xi)\left[-\ln|\xi-x| - \frac{\pi\,\mathrm{tg}\,\pi\mu}{2}\,\mathrm{sgn}(\xi-x)\right]d\xi = \pi f(x) -$$

$$- P^\circ[\ln\lambda + F_{11}(0)] - iQ^\circ \qquad (|x| \leqslant 1) \tag{1.15}$$

Here, respectively, for the cases

a) $\varphi^\circ(\xi) = \varphi_1^\circ(\xi) + i\varphi_2^\circ(\xi), \quad f(x) = f_1(x) + if_2(x), \qquad P^\circ = P_1^\circ + iP_2^\circ$ (1.16)

$$\mu = \frac{1}{2\pi i}\ln\frac{1+\varepsilon}{1-\varepsilon}, \qquad P_j^\circ = \int_{-1}^{1}\varphi_j^\circ(\xi)\,d\xi, \qquad Q^\circ = P_2^\circ[F_{22}(0) - F_{11}(0)]$$

b) $\varphi^\circ(\xi) = \varphi_1^\circ(\xi), \quad f(x) = f_1(x), \qquad P^\circ = P_1^\circ$

$$Q^\circ = 0, \qquad \mu = \frac{1}{2\pi i}\ln\frac{1+ik\varepsilon}{1-ik\varepsilon} \tag{1.17}$$

c) $\varphi^\circ(\xi) = \varphi_1^\circ(\xi), \quad f(x) = f_1(x), \quad P^\circ = P_1^\circ, \quad Q^\circ = 0, \ \mu = 0$ (1.18)

Differentiating both sides of (1.15) with respect to x we have

$$\int_{-1}^{1}\frac{\varphi^\circ(\xi)\,d\xi}{\xi - x} + \pi\,\mathrm{tg}\,\pi\mu\varphi^\circ(x) = \pi f'(x) \qquad (|x| \leqslant 1) \tag{1.19}$$

We have here utilized the relationship

$$1/_2\,(\mathrm{sgn}\,x)' = \delta(x) \tag{1.20}$$

and the known properties of the delta function $\delta(x)$.

Under the assumptions made on the function $f_1(x)$ the solution of the integral equation (1.19) can be obtained by solving the appropriate Riemann problem [3], and it will be

$$\varphi^\circ(x) = \frac{\cos\pi\mu}{\pi X(x)}\left[P^\circ - \cos\pi\mu\int_{-1}^{1}\frac{f'(\xi)\,X(\xi)\,d\xi}{\xi - x}\right] + \frac{1}{2}\sin 2\pi\mu f'(x) \tag{1.21}$$

$$X(x) = (1+x)^{1/2+\mu}(1-x)^{1/2-\mu} \tag{1.22}$$

In order for the expression of $\varphi^\circ(x)$ in the form (1.21) to satisfy the integral equation (1,15) also, it is necessary to define the quantity P° appropriately. To do this let us apply the following artificial procedure.

Let us note that the following relationship holds [1]:

$$\int_{-1}^{1}\left[-\ln|\xi - x| + \frac{\pi\,\mathrm{tg}\,\pi\mu}{2}\,\mathrm{sgn}(\xi - x)\right]\frac{d\xi}{Y(\xi)} = \frac{\pi}{\cos\pi\mu}D_\mu \tag{1.23}$$

$$Y(x) = (1+x)^{1/2-\mu}(1-x)^{1/2+\mu}, \qquad D_\mu = -[\ln 2 + C + 0.5\psi(1/_2 + \mu) + $$
$$+ 0.5\psi(1/_2 - \mu)]$$

Here $\psi(x)$ is the Euler Psi-function, and C the Euler constant.

Let us multiply both sides of the integral equation (1.15) by $Y^{-1}(x)$, and let us integrate with respect to x between the limits -1 and 1. Then trasposing the integrals in the left side of the relationship obtained and taking account of (1.23) and

$$\int_{-1}^{1}\frac{d\xi}{Y(\xi)} = \frac{\pi}{\cos\pi\mu} \tag{1.24}$$

we will have

$$P^\circ[\ln\lambda + F_{11}(0) + D_\mu] + iQ^\circ = \cos\pi\mu\int_{-1}^{1}\frac{f(x)\,dx}{Y(x)} \tag{1.25}$$

For the considered cases we have from (1.25)

$$\text{a)}\ P_1{}^\circ = \cos\pi\mu\,[\ln\lambda + F_{11}(0) + D_\mu]^{-1}\,\mathrm{Re}\int_{-1}^{1}\frac{f(x)\,dx}{Y(x)} \tag{1.26}$$

$$P_2{}^\circ = \cos\pi\mu\,[\ln\lambda + F_{22}(0) + D_\mu]^{-1}\,\mathrm{Im}\int_{-1}^{1}\frac{f(x)\,dx}{Y(x)}$$

$$\text{b)}\ P_1{}^\circ = \cos\pi\mu\,[\ln\lambda + F_{11}(0) + D_\mu]^{-1}\int_{-1}^{1}\frac{f_1(x)\,dx}{Y(x)} \tag{1.27}$$

$$\text{c)}\ P_1{}^\circ = [\ln 2\lambda + F_{11}(0)]^{-1}\int_{-1}^{1}\frac{f_1(x)\,dx}{\sqrt{1-x^2}} \tag{1.28}$$

In obtaining (1.28) it has been taken into account that $\psi\,(^1/_2) = -C - 2\ln 2$. Let us present some other relationships which shall be needed later.
It has been shown in [1] that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\xi^m X(\xi)\,d\xi}{\xi - x} = \frac{x^m X(x)}{\operatorname{ctg}\pi\mu} - \frac{P_{m+1}(x)}{\cos\pi\mu} \qquad (|x| \leqslant 1)$$

$$P_k(x) = \sum_{n=0}^{k} a_n(\mu)\,x^{k-n}, \qquad a_n(\mu) = \sum_{r=0}^{n}\frac{(-1)^r(-^1/_2-\mu)_r(\mu-{}^1/_2)_{n-r}}{(n-r)!\,r!} \tag{1.29}$$

$$(z)_n = z(z+1)\ldots(z+n-1), \quad (z)_0 = 1$$

If $f(x) \in H_n{}^\alpha(-\beta, \beta)$, $n \geqslant 0$, $0 < \alpha \leqslant 1$, then it can be shown that

$$\left|f(x) - \sum_{k=0}^{n}\frac{(x-\xi)^k}{k!}f^{(k)}(\xi)\right| \leqslant A\,|x-\xi|^{\alpha+n} \tag{1.30}$$

for any x and $\xi \in [-\beta, \beta]$, $A = \mathrm{const} > 0$. If the function $f(x)$ is even, then we obtain by taking x^2, ξ^2 as new variables and utilizing (1.30)

$$\left|f(x) - \sum_{k=0}^{n}\frac{(x^2-\xi^2)^k}{(2k)!!}\,[L^k f(\xi)]\right| \leqslant B_+|x-\xi|^{\alpha+n} \qquad \left(L = \frac{1}{\xi}\frac{d}{d\xi}\right) \tag{1.31}$$

for any x and $\xi \in [-\beta, \beta]$, $B_+ = \mathrm{const} > 0$. Analogously, for odd $f(x)$ we have

$$\left|f(x) - x\sum_{k=0}^{n}\frac{(x^2-\xi^2)^k}{(2k)!!}\,[L^k f(\xi)\,\xi^{-1}]\right| \leqslant B_-|x-\xi|^{\alpha+n} \tag{1.32}$$

for any $x \in [-\beta, \beta]$ and $0 < \varepsilon \leqslant |\xi| \leqslant \beta$, $B_- = \mathrm{const} > 0$.

Lemma 1. Let $f(z) \in H_n{}^\lambda(L)$, $n \geqslant 0$, $0 < \lambda \leqslant 1$ be an L-smooth closed contour in the complex z-plane. Then the function

$$F(z) = \frac{1}{2\pi i}\int_L\frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad (z \in L) \tag{1.33}$$

also belongs to $H_n{}^\mu(L)$, and $\mu = \lambda$ if $\lambda < 1$ and $\mu = 1 - \varepsilon$, where ε is an arbitrarily small positive number if $\lambda = 1$.

The lemma follows easily from the results in sections 4.4 and 5.1 of the monograph [3].

2. Investigation of the structure of the solution (1.5) of the integral equations (1.1) – (1.3). Let us first study the structure of the solution $\varphi^{\circ}(x)$ of the integral equation (1.15).

Theorem 1. If $f(x) \in H_{n+1}^{\alpha}(-1, 1)$, $n \geqslant 0$, $0 < \alpha \leqslant 1$, the function $\varphi^{\circ}(x)$ has the form

$$\varphi^{\circ}(x) = \omega^{\circ}(x) X^{-1}(x) \tag{2.1}$$

Here $\omega^{\circ}(x) \in H_n^{\gamma}(-1, 1)$, where $\gamma = \alpha$ if $\alpha < 1$, and $\gamma = 1 - \varepsilon$ if $\alpha = 1$, and ε is an arbitrarily small positive number.

Proof. Let us first consider the case of odd $f(x)$. Using (1.29), we represent (1.21) for $\varphi^{\circ}(x)$ as

$$\varphi^{\circ}(x) = \frac{\cos\pi\mu}{\pi X(x)}\left[P^{\circ} - \cos\pi\mu \int_{-1}^{1} \frac{\Phi_n(\xi) X(\xi)\, d\xi}{\xi - x} + Q_{2n+1}(x)\right] + \frac{1}{2}\sin 2\pi\mu\Phi_n(x)$$

$$\Phi_n(x) = f'(x) - \sum_{k=0}^{n} \frac{(-1)^k (1 - x^2)^k}{(2k)!!}[L^k f'(\xi)]_{\xi=1} \tag{2.2}$$

Here $Q_{2n+1}(x)$ is a polynomial of degree $2n + 1$.

It is easy to see that $\Phi_n(x) \in H_n^{\alpha}(-1, 1)$. Moreover, it follows from (1.31) that the function $\Phi_n(x)$ behaves as $(1 \mp x)^{n+\alpha}$ in the neighborhood of the points $x = \pm 1$. There remains to show that the integral

$$J(x) = \int_{-1}^{1} \frac{\Phi_n(\xi) X(\xi)\, d\xi}{\xi - x} \qquad (|x| \leqslant 1) \tag{2.3}$$

belongs to $H_n^{\gamma}(-1, 1)$ as a function of x.

Let us consider the auxiliary integral

$$J^*(z) = \int_{L} \frac{\Phi_n^*(\zeta)\, d\zeta}{\zeta - z} \quad (z \in L), \qquad \Phi_n^*(z) = \begin{cases} 0, & z \in L'' \\ \Phi_n(x) X(x), & z \in L' \end{cases} \tag{2.4}$$

Here L' is a segment of the real axis $|\xi| \leqslant 1$ in the complex plane, and L'' is its smooth closure (see Fig. 1).

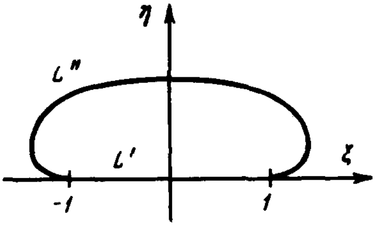

Fig. 1

Taking account of the properties of $\Phi_n(x)$ it is easy to show that $\Phi_n^*(z) \in H_n^{\alpha}(L)$. Then on the basis of Lemma 1 it can be concluded that

$$J^*(z) \in H_n^{\gamma}(L). \tag{2.5}$$

Now noting that $J^*(z)$ agrees with the integral $J(x)$ on the segment L', we are assured of the validity of the theorem for the case of an odd function $f(x)$. The case of even $f(x)$ is considered analogously by relying on the relationship (1.32).

Corollary 1. If $f(x) \in H_1^{\alpha}(-1, 1)$, $0 < \alpha \leqslant 1$, the function $\varphi^{\circ}(x) \in L_p(-1, 1)$, $1 \leqslant p \leqslant \varkappa < 2$. Here $\varkappa = 2 - \varepsilon$ for cases (a) and (c), and $\varkappa = 2(1 + 2\mu)^{-1} - \varepsilon$ for case (b), (ε is an arbitrarily small positive number).

Let us now turn to a study of the structure of the functions $\varphi_j^*(\xi)$. We shall hence later consider two fundamental versions:

$$\text{A)}\ F_{jj}(t)=c_{jj}|t|+G_{jj}(t),\qquad F_{12}(t)=b_{12}t\ln|t|+G_{12}(t)\tag{2.6}$$

$$G_{jl}(t)\in H_1^{\nu}(-2/\lambda,\ 2/\lambda),\qquad 0<\nu\leqslant 1,\ \lambda>0$$

$$\text{B)}\ F_{jl}(t)\in H_{k+1}^{\nu}(-2/\lambda,\ 2/\lambda),\qquad 0<\nu\leqslant 1,\qquad \lambda>0,k\geqslant 0\tag{2.7}$$

We moreover assume that the solutions $\varphi_j^*(\xi)$ of the integral equations (1.9)–(1.14) exist in $L_q(-1,\ 1)$, $1<q\leqslant\varkappa$.

Let us note that the version A is encountered in mixed problems for domains with circular boundaries, for instance, in contact problems of elasticity theory for an infinite circular tube.

The version B is encountered in studying mixed problems for strips, wedges, etc.

Let us first elucidate the properties of functions $f_j^*(x)$ of the form (1.10), (1.12), (1.14) for the versions mentioned.

Taking account of (2.6), let us rewrite (1.10) as

$$f_j^*(x)=-\frac{c_{jj}}{\pi}\int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\frac{|\xi-x|}{\lambda}d\xi\pm\frac{\varepsilon b_{12}}{\pi}\int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\times$$

$$\times\frac{(\xi-x)}{\lambda}\ln\frac{|\xi-x|}{\lambda}d\xi-\frac{1}{\pi}\int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\left[G_{jj}\left(\frac{\xi-x}{\lambda}\right)-G_{jj}(0)\right]d\xi\pm$$

$$\pm\frac{\varepsilon}{\pi}\int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\,G_{12}\left(\frac{\xi-x}{\lambda}\right)d\xi\tag{2.8}$$

On the basis of the assumption A relative to the properties of the functions $\varphi_j^*(\xi)$ and $G_{jl}(t)$, and also taking account of the properties of the functions $\varphi_j^\circ(\xi)$, it is not at all difficult to show that the third and fourth integrals in (2.8) belong, as functions of x, to $H_1^{\nu}(-1,\ 1)$. In order to investigate the first two integrals in (2.8), let us differentiate them with respect to x. We have

$$\int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\operatorname{sgn}\left(\frac{\xi-x}{\lambda}\right)d\xi,\qquad \int_{-1}^{1}[\varphi_j^\circ(\xi)+\varphi_j^*(\xi)]\left[\ln\frac{|\xi-x|}{\lambda}+1\right]d\xi\tag{2.9}$$

Utilizing the Hölder inequality and the properties of the functions $\varphi_j^\circ(\xi)$, $\varphi_j^*(\xi)$, it can be proved that the first integral in (2.9) belongs, as a function of x, to $H_0^{r}(-1,\ 1)$, and the second to $H_0^{r-\varepsilon}(-1,\ 1)$. Here

$$r=\operatorname{In}f\left(\frac{p-1}{p},\ \frac{q-1}{q}\right)$$

and ε is an arbitrarily small positive number. Therefore, the first integral in (2.8) belongs to $H_1^{r}(-1,\ 1)$, and the second to $H_1^{r-\varepsilon}(-1,1)$.

Hence, for the version A the functions $f_j^*(x)\in H_1^{s}(-1,\ 1)$, where $s=\operatorname{In}f\,(r-\varepsilon,\ \nu)$ for cases (a), (b), and

$$s=\operatorname{In}f\,(r,\ \nu)$$

for case (c).

On the basis of the assumptions made and the properties of $\varphi_j^\circ(\xi)$, we easily find for version B that

$$f_j^*(x) \in H_{k+1}^{\nu}(-1, 1)$$

for all the cases considered.

The following theorems about $\varphi_j^*(x)$ can be formulated.

Theorem 2. If $f(x) \in H_1^{\alpha}(-1, 1)$, $0 < \alpha \leqslant 1$, and if the assumptions A are valid, the function $\varphi^*(x)$ has the form

$$\varphi^*(x) = \omega^*(x) X^{-1}(x) \tag{2.10}$$

where $\omega^*(x) \in H_0^s(-1, 1)$.

Theorem 3. If $f(x) \in H_1^{\alpha}(-1, 1)$, $0 < \alpha \leqslant 1$, and if assumptions B are valid, the function $\varphi^*(x)$ has the form (2.10), where $\omega^*(x) \in H_{k+1}^{l}(-1, 1)$ and $l = \nu$ if $\nu < 1$, and $l = 1 - \varepsilon$ if $\nu = 1$, (ε is an arbitrarily small positive number) (*).

Here $\varphi^*(x) = \varphi_1^*(x) + i\varphi_2^*(x)$ for case (a), and $\varphi^*(x) = \varphi_1^*(x)$ for cases (b) and (c). The proof of Theorems 2 and 3 follows easily from Theorem 1 and the above-mentioned properties of the functions $f_j^*(x)$.

3. On effective asymptotic methods of solving the integral equations (1.1)—(1.3). Effective asymptotic methods of solving integral equations of the type (1.3) are elucidated in [4]. Some of these methods can be used, without essential changes, for the approximate solution of integral equations of types (1.1), (1.2) as well.

Asymptotic solutions of (1.1), (1.2) can be obtained by a method analogous to that described in [4], Sect. 2 for the large values of the parameter λ. Let us demonstrate this by the example of Eqs. (1.2), (1.4).

Under the assumptions A, the integral equation (1.2) can be represented, on the basis of the relationships (1.21), (1.27), as an equivalent integral equation of the second kind in $L_p(-1, 1)$ for $\lambda > 0$:

$$\begin{aligned}\varphi_1(x) = \frac{\cos \pi\mu}{\pi X(x)} \Big\{ P_1 - \cos \pi\mu \int_{-1}^{1} \frac{f_1'(t) X(t)\, dt}{t - x} - \frac{\cos \pi\mu}{\pi\lambda} \int_{-1}^{1} \frac{X(t)\, dt}{t - x} \int_{-1}^{1} \varphi_1(\xi) \times \\ \times \Big[F_{11}'\Big(\frac{\xi - t}{\lambda}\Big) - \operatorname{tg} \pi\mu F_{12}'\Big(\frac{\xi - t}{\lambda}\Big)\Big] d\xi \Big\} + \frac{1}{2} \sin 2\pi\mu f_1'(x) + \\ + \frac{1}{2\pi\lambda} \sin 2\pi\mu \int_{-1}^{1} \varphi_1(\xi) \Big[F_{11}'\Big(\frac{\xi - x}{\lambda}\Big) - \operatorname{tg} \pi\mu F_{12}'\Big(\frac{\xi - x}{\lambda}\Big)\Big] d\xi\end{aligned} \tag{3.1}$$

under the additional condition

$$\begin{aligned}P_1 = \int_{-1}^{1} \varphi_1(\xi)\, d\xi = \cos \pi\mu (\ln \lambda + D\mu)^{-1} \Big\{ \int_{-1}^{1} \frac{f_1(t)\, dt}{Y(t)} - \\ - \frac{1}{\pi} \int_{-1}^{1} \frac{dt}{Y(t)} \int_{-1}^{1} \varphi_1(\xi) \Big[F_{11}\Big(\frac{\xi - t}{\lambda}\Big) - \operatorname{tg} \pi\mu F_{12}\Big(\frac{\xi - t}{\lambda}\Big)\Big] d\xi \Big\}\end{aligned} \tag{3.2}$$

Now, let us assume that the following expansions

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*) It can be shown that the results of Theorems 2 and 3 are conserved even if the function $f'(x)$ has a singularity of the type $(1 - x^2)^{-\theta}$, $0 < \theta < 1/2$ at the points $x = \pm 1$.

$$F_{jj}(t)=\sum_{k=0}^{\infty}(a_{jjk}+b_{jjk}|t|+c_{jjk}\ln|t|)\,t^{2k}$$
$$(c_{jl_0}=0) \qquad (3.3)$$
$$F_{12}(t)=\sum_{k=0}^{\infty}(a_{12k}+b_{12k}t\ln|t|+c_{12k}\operatorname{sgn}t)\,t^{2k}$$

which converge uniformly for all $|t|<\rho$ are effective for the function $F_{jl}(t)$ (*). Then all the results based on (3.3) will be satisfiable for at least all $\lambda>2\rho^{-1}$.

Let us substitute (3.3) into (3.1), and let us seek its solution as

$$\varphi_1(x)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varphi_{1mn}(x)\,\lambda^{-m}\ln^n\lambda \qquad (3.4)$$

Equating coefficients in the right and left sides of (3.1) for identical powers of λ^{-1} and $\ln\lambda$, we obtain an infinite system of relationships for the sequential determination of the functions $\varphi_{1mn}(x)$, which we shall not present in the interests of brevity.

If

$$b_{jlk}=c_{jlk}=0 \qquad (3.5)$$

in (3.3), and the function $f_1(x)$ is a polynomial, then in determining $\varphi_{1mn}(x)$ from the above-mentioned relationships all the quadratures are taken in closed form (see [1], formulas (3.1), (3.3), for example). In the general case, some of the first few functions $\varphi_{1mn}(x)$ can be determined approximately, just as has been done in [5].

After the required number of functions $\varphi_{1mn}(x)$ have been found (depending on the desired accuracy of the asymptotic solution (3.4)), the quantity P_1 is determined by formula (3.2).

For small values of the parameter λ the construction of the asymptotic solution of the integral equation (1.2) by the method elucidated in [4], Sect. 4, causes no special difficulties. Hence, we elucidate just the scheme for constructing an asymptotic solution for small λ for the system of integral equations (1.1).

Let us rewrite the system (1.1) as

$$\int_{-1}^{1}\varphi(\xi)\,K\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi g(x) \qquad (|x|\leqslant 1)$$
$$\varphi(x)=\varphi_1(x)+i\varphi_2(x), \qquad K(t)=K_{11}(t)+i\varepsilon K_{12}(t) \qquad (3.6)$$
$$g(x)=f(x)+\frac{1}{\pi i}\int_{-1}^{1}\varphi_2(\xi)\,M\left(\frac{\xi-x}{\lambda}\right)d\xi, \qquad M(t)=K_{22}(t)-K_{11}(t)$$
$$f(x)=f_1(x)+if_2(x)$$

Now it is convenient to solve the system (3.6) by successive approximations according to the scheme $\quad \Phi_n(x)=\Phi_{1n}(x)+i\Phi_{2n}(x)\to\varphi(x), \qquad n\to\infty \qquad (3.7)$

$$\int_{-1}^{1}\Phi_n(\xi)\,K\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi g_n(x) \qquad (|x|\leqslant 1)$$

*) Such expansions are obtained for $F_{jl}(t)$ in all mixed problems for domains such as an infinite circular tube, strip, wedge, etc.

$$g_n(x) = f(x) + \frac{1}{\pi i} \int_{-1}^{1} \Phi_{2,\,n-1}(\xi)\, M\left(\frac{\xi - x}{\lambda}\right) d\xi, \qquad g_0(x) = f(x) \tag{3.8}$$

where the asymptotic solution, for small λ, of the integral equation (3.7) can be constructed roughly in the same manner as has been described in [4], Sect. 4.

Indeed, let us represent (3.7) as a system of three integral equations

$$\int_{-1}^{\infty} \psi_{1n}\left(\frac{1+\xi}{\lambda}\right) K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi g_n(x) + \int_{-\infty}^{-1} \left[\psi_{2n}\left(\frac{1-\xi}{\lambda}\right) - v_n(\xi)\right] K\left(\frac{\xi - x}{\lambda}\right) d\xi$$
$$(-1 \leqslant x < \infty) \tag{3.9}$$
$$\int_{-\infty}^{1} \psi_{2n}\left(\frac{1-\xi}{\lambda}\right) K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi g_n(x) + \int_{1}^{\infty} \left[\psi_{1n}\left(\frac{1+\xi}{\lambda}\right) - v_n(\xi)\right] K\left(\frac{\xi - x}{\lambda}\right) d\xi$$
$$(-\infty < x \leqslant 1)$$

$$\int_{-\infty}^{\infty} v_n(\xi)\, K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi g_n(x) \qquad (-\infty < x < \infty) \tag{3.10}$$

which are equivalent to (3.7) under the condition

$$\Phi_n(\xi) = \psi_{1n}\left(\frac{1+\xi}{\lambda}\right) + \psi_{2n}\left(\frac{1-\xi}{\lambda}\right) - v_n(\xi) \qquad (|\xi| \leqslant 1) \tag{3.11}$$

The function $g_n(x)$ is continued with the conservation of sufficient smoothness into the intervals $-\infty < x < -1$, $1 < x < \infty$. Moreover, if

$$K(t) \sim e^{-\varkappa |t|}, \qquad |t| \to \infty \tag{3.12}$$

it is necessary that

$$\int_{-\infty}^{\infty} e^{-\varkappa^* |x|} f(x)\, dx < \infty, \qquad \varkappa^* < \varkappa \tag{3.13}$$

The solution of the integral equation (3.10) is easily found by applying the convolution theorem for Fourier transforms. By a change of variable the integral equations (3.9) are reduced to the following:

$$\int_{0}^{\infty} \psi_{1n}(\tau)\, K(\tau - t)\, d\tau = \pi g_n(\lambda t - 1) +$$
$$+ \int_{2/\lambda}^{\infty} [\psi_{2n}(\tau) - v_n(1 - \lambda\tau)]\, K\left(\frac{2}{\lambda} - \tau - t\right) d\tau \quad (0 \leqslant t < \infty) \tag{3.14}$$
$$\int_{0}^{\infty} \psi_{2n}(\tau)\, K(\tau - t)\, d\tau = \pi g_n(1 - \lambda t) +$$
$$+ \int_{2/\lambda}^{\infty} [\psi_{1n}(\tau) - v_n(\lambda\tau - 1)]\, K\left(\tau + t - \frac{2}{\lambda}\right) d\tau$$

It is natural to solve the system of integral equations (3.14) for small λ by successive approximations according to the scheme

$$\Psi_{1nm} \to \psi_{1n}, \qquad \Psi_{2nm} \to \psi_{2n}, \qquad m \to \infty$$

$$\int_{0}^{\infty} \Psi_{1n0}(\tau)\, K(\tau - t)\, d\tau = \pi g_n(\lambda t - 1), \qquad \int_{0}^{\infty} \Psi_{2n0}(\tau)\, K(t - \tau)\, d\tau = \pi g_n(1 - \lambda t)$$

(cont.)

$$\int_0^\infty \Psi_{1nm}(\tau)\,K(\tau-t)\,d\tau = \pi g_n(\lambda t-1) + \int_{2/\lambda}^\infty [\Psi_{2n,\,m-1}(\tau) - v_n(1-\lambda\tau)] \times$$
$$\times K(2/\lambda-\tau-t)\,d\tau \tag{3.15}$$
$$\int_0^\infty \Psi_{2nm}(\tau)\,K(t-\tau)\,d\tau = \pi g_n(1-\lambda t) + \int_{2/\lambda}^\infty [\Psi_{1n,\,m-1}(\tau) - v_n(\lambda\tau-1)] \times$$
$$\times K(\tau+t-2/\lambda)\,d\tau$$
$$(m \geqslant 1,\ 0 \leqslant t < \infty)$$

At each step it is hence necessary to find the solution of Wiener-Hopf integral equations with the same kernels but with different right-hand sides.

In order to obtain solutions of the mentioned Wiener-Hopf integral equations which can be practically applied, the Koiter method of approximate factorization [6] must be utilized. It is generally possible to approximate a kernel $K(t)$ of the form (3.6) such that all its fundamental properties are conserved, and the final solution is expressed in terms of tabulated functions.

Let us demonstrate this by the example of impressing a stamp in an elastic layer of thickness h clamped rigidly to a base in the presence of complete adhesion along the contact line $-a \leqslant x \leqslant a$, $\lambda = h/a$, $\varepsilon = (1-2\sigma)\,[2\,(1-\sigma)]^{-1}$ where σ is the Poisson's ratio.

For this problem [1]

$$K_{jj}(t) = \int_0^\infty \frac{L_{jj}(u)}{u}\cos ut\,du, \qquad K_{12}(t) = \int_0^\infty \frac{L_{12}(u)}{u}\sin ut\,du \qquad (j=1,\,2) \tag{3.16}$$

$$L_{jj}(u) = \frac{2\varkappa\,\mathrm{sh}\,2u + 4\,(-1)^j u}{2\varkappa\,\mathrm{ch}\,2u + \varkappa^2 + 1 + 4u^2}, \qquad L_{12}(u) = \frac{2\varkappa\,(\mathrm{ch}\,2u - 1) - 8\,(\varkappa-1)^{-1}u^2}{2\varkappa\,\mathrm{ch}\,2u + \varkappa^2 + 1 + 4u^2}$$
$$(\varkappa = 3-4\sigma)$$

We then have

$$K(t) = \frac{1}{2}\int_{-\infty}^\infty \frac{L(u)}{u}e^{iut}\,du, \qquad L(u) = L_{11}(u) + \varepsilon L_{12}(u) \tag{3.17}$$

It is easily established on the basis of (3.16) that

$$\begin{aligned} &u^{-1}L(u) \sim |u|^{-1}(1+\varepsilon\,\mathrm{sgn}\,u), && u \to \pm\infty \\ &u^{-1}L(u) \sim A_0 + A_1 u, && u \to 0 \\ &A_0 = 2\varepsilon\,(1-\varepsilon), \qquad A_1 = 2\varepsilon\,(1-\varepsilon) - (1-\varepsilon)^2 \end{aligned} \tag{3.18}$$

Let us now find the solution of the Wiener-Hopf integral equation

$$\int_0^\infty \psi(\tau)\,K(\tau-t)\,d\tau = \pi p(t), \qquad 0 \leqslant t < \infty \tag{3.19}$$

with the kernel (3.17). In order to do the approximate factorization, we first approximate the function $L(u)\,u^{-1}$ by the expression

$$\frac{L^*(u)}{u} = \frac{B\,(u+iD)^{1/2-i\beta}\,(u-iD)^{1/2-i\beta}}{u^2-E^2} = \frac{B\,\sqrt{u^2+D^2}\,\exp\,[-2\beta\,\mathrm{arctg}\,(Du^{-1})]}{u^2+E^2} \tag{3.20}$$

We select the constants B, D, E and β in (3.20) so that the behavior of the approximating function $u^{-1}L^*(u)$ would agree with the behavior of the function $u^{-1}L(u)$ defined by (3.18), at zero and infinity. After simple computations we find

$$B = 1 + \varepsilon, \quad D = \frac{2\beta A_0}{A_1}, \quad E^2 = \frac{2\beta\sqrt{1-\varepsilon^2}}{A_1}, \quad \beta = \frac{1}{2\pi}\ln\frac{1+\varepsilon}{1-\varepsilon} = \mu i \quad (3.21)$$

Furthermore, let us examine the case $p(t) \equiv p = \mathrm{const}$. Solving the integral equation (3.19) with kernel (3.17), (3.20) by a known scheme [6], we obtain

$$\psi(t) = \frac{pEe^{\pi\beta}}{BD^{1/2-i\beta}\,\Gamma(1/2+i\beta)}\left[\frac{e^{-Dt}}{t^{1/2-i\beta}} + ED^{-1/2-i\beta}\gamma(1/2+i\beta,\, Dt)\right] \quad (3.22)$$

Here $\gamma(\alpha, x)$ is the incomplete Gamma function

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}\,dt$$

for which there is a table in [7].

Let us note that the integral equation (3.19) corresponds to equations (3.15) for $\Psi_{1nm}(\tau)$. Taking this into account, substituting the expressions $t = (1+x)\lambda^{-1}$ and $\beta = \mu i$ into (3.22), we see that the approximate solution obtained for (3.19) possesses a singularity of the form $(1+x)^{-1/2-\mu}$ at the point $x = -1$. This corresponds completely to the facts established in the first two sections relative to singulatities of the solutions of mixed problems with adhesion. Let us note, finally, that a more exact approximation than (3.20) is obtained by multiplying $u^{-1}L^*(u)$ by an appropriate rational function.

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